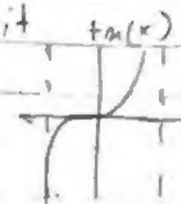
Last Time: Space Curves

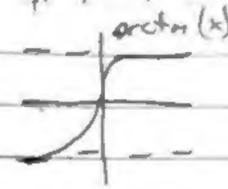
$$\vec{r} : \underbrace{I}_{\text{Interval}} \to \mathbb{R}^n$$

Recall: Limit of space curve is the component-wise limit



Ex: Compute $\lim_{t\to\infty} \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle$

Soln: $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{1+t^2}{1-t^2} = \lim_{t\to\infty} \frac{\frac{1}{t^2}+1}{\frac{1}{t^2}-1} = \frac{0+1}{0-1} = -1$



$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$

$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \frac{1-e^{-2t}}{t} = 0$

Hence, $\lim_{t\to\infty} \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle = \left\langle -1, \frac{\pi}{2}, 0 \right\rangle$

Defn: A space curve $\vec{r}(t)$ is continuous at time $t=a$ if
$$\lim_{t\to\infty} \vec{r}(t) = \vec{r}(a)$$

"cts" = "Continuous"

NB: A curve is continuous at time $t=a$ if and only if each of its components is continuous at time $t=a$.

Ex: When is $\vec{r}(t) = \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle$ cts?

Soln: $x(t) = \frac{1+t^2}{1-t^2}$ is cts at t iff $t \neq \pm 1$, i.e. $t \in (-\infty,-1) \cup (-1,1) \cup (1,\infty)$

$y(t) = \arctan(t)$ is cts for $t \in (-\infty, \infty)$

$z(t) = \frac{1-e^{-2t}}{t}$ is cts on $t \in (-\infty, 0) \cup (0,\infty)$

$\therefore$ $\vec{r}(t)$ is cts for $t \in (-\infty,-1) \cup (-1,0) \cup (0,1) \cup (1,\infty)$

Derivatives

Defn. The derivative of space curve $\vec{r}(t)$ at the time $t=a$ is

$$\vec{r}\,'(a) = \frac{d\vec{r}}{dt}\Big|_{t=a} = \lim_{h\to 0} \frac{\vec{r}(a+h)-\vec{r}(a)}{h}$$

Ex: Compute $\vec{r}\,'(t)$ for $\vec{r}(t) = \langle t, t^3, \sqrt{t} \rangle$

Soln: $\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\langle t+h, (t+h)^3, \sqrt{t+h}\rangle\right)$

$= \lim_{h\to 0} \frac{1}{h} \langle h, 3t^2h+3th^2+h^3, \sqrt{t+h}-\sqrt{t} \rangle$

$= \lim_{h\to 0} \langle 1, 3t^2+3th+h^2, \frac{\sqrt{t+h}-\sqrt{t}}{h} \rangle$

$= \langle \lim_{h\to 0} 1, \lim_{h\to 0} 3t^2+3th+h^2, \lim_{h\to 0} \frac{\sqrt{t+h}-\sqrt{t}}{h} \rangle$

$= \langle 1, 3t^2, \frac{1}{2}t^{-\frac{1}{2}} \rangle$

because: $\lim_{h\to 0} \frac{\sqrt{t+h}-\sqrt{t}}{h} = \lim_{h\to 0} \left(\frac{\sqrt{t+h}-\sqrt{t}}{h} \cdot \frac{\sqrt{t+h}+\sqrt{t}}{\sqrt{t+h}+\sqrt{t}} \right) = \lim_{h\to 0} \frac{1}{\sqrt{t+h}+\sqrt{t}}$
$= \frac{1}{\sqrt{t}+\sqrt{t}} = \frac{1}{2\sqrt{t}}$

What's really going on $(n=2)$: $\vec{r}(t) = \langle x(t), y(t) \rangle$

I chose the wrong day to take notes

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$
$$= \lim_{h\to 0} \left\langle \frac{x(t+h)-x(t)}{h}, \frac{y(t+h)-y(t)}{h} \right\rangle$$
$$= \left\langle \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}, \lim_{h\to 0} \frac{y(t+h)-y(t)}{h} \right\rangle$$
$$= \langle x'(t), y'(t) \rangle$$

Point: The derivative is again a comparative operation

Prop (Properties of Space-Curve Derivative):
Let $\vec{r}(t)$, and $\vec{s}(t)$ be space curves in $\mathbb{R}^n$ and let $c(t)$ be a scalar function.

① $\frac{d}{dt}\left[\vec{r}(t)+\vec{s}(t)\right]=\vec{r}'(t)+\vec{s}'(t)$ Sum Rule

② $\frac{d}{dt}\left[c(t)\vec{r}(t)\right]=c'(t)\vec{r}(t)+c(t)\vec{r}'(t)$ Scalar Product Rule

③ $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{s}(t)\right]=\vec{r}'(t)\cdot\vec{s}(t)+\vec{r}\cdot\vec{s}'(t)$ Dot Product Rule

$(n=3)$ ④ $\frac{d}{dt}\left[\vec{r}(t)\times\vec{s}(t)\right]=\vec{r}'(t)\times\vec{s}(t)+\vec{r}(t)\times\vec{s}'(t)$ Cross Product Rule

⑤ $\frac{d}{dt}\left[\vec{r}(c(t))\right]=\vec{r}'(c(t))\,c'(t)$ Chain Rule

Exercise: Verify each of the properties for space curves in $\mathbb{R}^3$

Terminology: The tangent vector to space curve $\vec{r}(t)$ a time $t=a$ is $\vec{r}'(a)$.

The unit tangent vector at $t=a$ is $\frac{r'(t)}{r'(a)}$.

The speed of $\vec{r}(t)$ at $t=a$ is $|\vec{r}'(a)|$.

Exercise: Prove that if $\vec{r}(t)$ has constant speed, then $\vec{r}(t)$ is orthogonal to $\vec{r}(t)$ for all t.

Integrals:
Defn. The definite integral of space curve from $t=a$ to b is

$$\int_{t=a}^{b}\vec{r}(t)\,dt=\int_{t=a}^{b}\langle x(t),y(t),z(t)\rangle\,dt=\left\langle\int_{t=a}^{b}x(t)\,dt,\int_{t=a}^{b}y(t)\,dt,\int_{t=a}^{b}z(t)\,dt\right\rangle$$

Interpretation: the displacement vector (of calc. I)

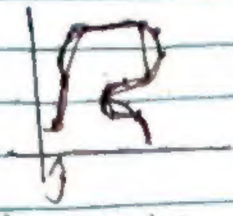
Arc Length

The arc length of space curve $\vec{r}(t)$ from $t=a$ to b is

$$s=\int_{t=a}^{b}|\vec{r}'(t)|\,dt$$

limit of secant lines = tangent lines

bad vs. better



"Piecewise-linear approx of curve"

↳ approx. length